1 Problem

We are going to prove Wallis's formula for odd powers with $n \geq 3$:

$$\int_0^{\pi/2} \cos^n x \; \mathrm{d}x = \frac{2}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{n-1}{n}$$

We are going to use the following formula (problem 80):

$$\int \cos^n x \; \mathrm{d}x = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \; \mathrm{d}x$$

2 Proof

We are going to do this in two steps (this is known as proof by induction):

1. Prove that the formula is true for $n = 3$
2. Prove that if the formula is true for $n - 2$, it is also true for n.

2.1 Proof for $n = 3$

$$\begin{aligned}
\int_0^{\pi/2} \cos^3 x \; \mathrm{d}x &= \left.\frac{\cos^2 x \sin x}{3}\right|_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos x \; \mathrm{d}x \\
&= \left.\frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x\right|_0^{\pi/2} \\
&= \left(\frac{\cos^2 \frac{\pi}{2} \sin \frac{\pi}{2}}{3} + \frac{2}{3} \sin \frac{\pi}{2}\right) - \left(\frac{\cos^2 0 \sin 0}{3} + \frac{2}{3} \sin 0\right) \\
&= \left(\frac{0 \cdot 1}{3} + \frac{2}{3} \cdot 1\right) - \left(\frac{1 \cdot 0}{3} + \frac{2}{3} \cdot 0\right) \\
&= \frac{2}{3} - 0 \\
&= \frac{2}{3}
\end{aligned}$$

2.2 Prove if true for $n - 2$ then true for n.

Assuming the formula is true for $n - 2$, that means

$$\int_0^{\pi/2} \cos^{n-2} x \; \mathrm{d}x = \frac{2}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{n-3}{n-2}.$$

Now we want to find $\int_0^{\pi/2} \cos^{n-2} x \, \mathrm{d}x$. Using the formula from problem 80, we get:

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x \, \mathrm{d}x &= \left.\frac{\cos^{n-1} x \sin x}{n}\right|_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, \mathrm{d}x \\
&= \left(\frac{\cos^{n-1} \frac{\pi}{2} \sin \frac{\pi}{2}}{n}\right) - \left(\frac{\cos^{n-1} 0 \sin 0}{n}\right) + \frac{n-1}{n} \frac{2}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{n-3}{n-2} \\
&= \left(\frac{0 \cdot 1}{n}\right) - \left(\frac{1 \cdot 0}{n}\right) + \frac{2}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{n-3}{n-2} \frac{n-1}{n} \\
&= \frac{2}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{n-3}{n-2} \frac{n-1}{n}
\end{aligned}$$

This completes the proof because it means the formula is true for $n = 3$ by the first part. By the second part it is then true for $n = 5$, since it is true for $n = 5$, it is true for $n = 7$, and so on.

$\square$